

HOMOLOGICAL PISOT SUBSTITUTIONS AND EXACT REGULARITY

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ABSTRACT. We consider one-dimensional substitution tiling spaces where the dilatation (stretching factor) is a degree d Pisot number, and where the first rational Čech cohomology is d -dimensional. We construct examples of such “homological Pisot” substitutions that do not have pure discrete spectra. These examples are not unimodular, and we conjecture that the coincidence rank must always divide a power of the norm of the dilatation. To support this conjecture, we show that homological Pisot substitutions exhibit an Exact Regularity Property (ERP), in which the number of occurrences of a patch for a return length is governed strictly by the length. The ERP puts strong constraints on the measure of any cylinder set in the corresponding tiling space.

1. INTRODUCTION

Versions of the Pisot Conjecture occur in number theory (numeration systems, β -expansions), discrete geometry, dynamical systems (construction of Markov partitions, arithmetical coding of hyperbolic toral automorphisms), and physics (spectral properties of materials with quasi-periodic atomic structure) - see the survey [BS]. In all of these settings there is an underlying substitution and in this context a standard version of the conjecture is

Pisot Conjecture: The \mathbb{R} -action (tiling flow) on the tiling space associated with a one-dimensional substitution of unimodular and irreducible Pisot type has pure discrete spectrum.

It is known that any such action has a non-trivial discrete part in its spectrum, see [BT], and the conjecture is known to be true in the case of a substitution on two letters, see [BD2, HS]. One of the reasons for interest in this conjecture is that the \mathbb{R} -action on a substitution tiling space has a pure discrete spectrum (also known as pure point spectrum) if and only if a one-dimensional quasicrystal, whose atoms are

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arranged according to the pattern of any tiling in the tiling space, has pure discrete diffraction spectrum, see [Dw, LMS].

There are good reasons for replacing the “irreducible” assumption by a purely topological condition, and revisiting the “unimodular” assumption. Our preference for a topological condition begins with the observation that if two one-dimensional substitution tiling spaces are homeomorphic, then the tiling flow on one of them has pure discrete spectrum if and only if the tiling flow on the other does. (This is a consequence of the rigidity result of [BSw].) There are many ways of rewriting, or otherwise re-presenting, a substitution without altering the topology (or even dynamics) of the tiling space, but these rewritings generally do not preserve irreducibility. As for unimodularity, there are many instances where pure discreteness is known to hold when the Pisot dilatation has norm other than ± 1 (in particular, for a large family of substitutions associated with β -expansions - see [BBK]).

If the dilatation (also known as the stretching factor) of a primitive substitution is a Pisot number of algebraic degree d we will say that the substitution is *Pisot of degree d* . If in addition the dimension of the first rational Čech cohomology of the tiling space is d , then we say the substitution is *homological Pisot of degree d* . Let $p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ be the minimal polynomial of the dilatation. The *norm* of the dilatation is the product of the dilatation and its algebraic conjugates and equals $(-1)^d a_0$. We call a_0 the *constant coefficient* of the dilatation. The minimal polynomial $p(x)$ always divides the characteristic polynomial of the abelianization (also known as the substitution matrix). If the two polynomials are equal, then the substitution is said to be *irreducible*.

In this paper we study the consequences of a substitution being homological Pisot, including (but not limited to) the question of whether the tiling flow associated with such a substitution must have pure discrete spectrum. Note that the homological Pisot condition is neither stronger nor weaker than the irreducible Pisot condition. It is easy to construct homological Pisot substitutions that are not irreducible – just rewrite any homological Pisot substitution in terms of collared tiles. It is also easy to find examples of irreducible Pisot substitutions whose first cohomology has degree greater than d (e.g. the substitution, $1 \mapsto 21112$, $2 \mapsto 121$, is an irreducible degree 2 Pisot substitution with an asymptotic cycle, whose tiling space has 3-dimensional first cohomology).

Every one-dimensional substitution tiling space has a *maximal equicontinuous factor* consisting of a Kronecker flow on a torus or solenoid. The continuous map factoring the tiling flow onto its maximal equicontinuous factor is called *geometric realization* and also factors the substitution homeomorphism onto a hyperbolic automorphism of the torus or solenoid. Geometric realization has the following properties ([BK, BBK]): it is nontrivial if and only if the substitution is Pisot and, if the substitution is Pisot, then

geometric realization is boundedly finite-to-one and almost everywhere cr -to-one for some positive integer cr called the *coincidence rank* of the substitution. Furthermore, the tiling flow has pure discrete spectrum if and only if geometric realization is a.e. 1-to-1 (that is, if $cr = 1$), in which case geometric realization is a continuous measurable isomorphism from the tiling flow onto a Kronecker action.

If, in the Pisot Conjecture, the unimodular assumption (*i.e.*, $\text{norm} = \pm 1$) is dropped and “irreducible Pisot” is replaced by homological Pisot, one has a natural conjecture: *The tiling flow of a homological Pisot substitution has pure discrete spectrum.* This conjecture is **false**. In the last section of this paper we give several examples with coincidence rank 3 and norm divisible by 3. That the coincidence rank in these examples divides the norm is not a coincidence and we are led to the following conjecture.

Coincidence Rank Conjecture: The coincidence rank of a homological Pisot substitution divides a power of the norm. In particular, the tiling flow of a unimodular homological Pisot substitution has pure discrete spectrum.

Our results are summarized as follows:

Theorem 1. *There are homological Pisot substitutions of every algebraic degree whose tiling flows do not have pure discrete spectrum.*

Theorem 2. *The Coincidence Rank Conjecture is true if the degree of the substitution is one.*

The *tile lattice*, Γ , for the tiling space Ω_ϕ is the additive subgroup of \mathbb{R} generated by the lengths of the tiles of Ω_ϕ . A *return length* for the tiling space Ω_ϕ is a number $L > 0$ such that \mathbf{T} and $\mathbf{T} - L$ have nonempty intersection for some $\mathbf{T} \in \Omega_\phi$. The *return lattice* for Ω_ϕ is the subgroup of the tile lattice Γ generated by $\{\lambda^{-n}L : L \text{ is a return length and } n \in \mathbb{N}\} \cap \Gamma$, where λ is the dilatation of ϕ .¹

Definition 3. Let λ denote the dilatation of the substitution ϕ and let $L > 0$ be in the return lattice for Ω_ϕ . We say that the tiling space Ω_ϕ exhibits the Exact Regularity Property (ERP) if for each patch \mathbf{P} of any tiling in Ω_ϕ there is a length L' and a linear functional $\mathbf{N}_\mathbf{P} : \mathbb{Q}(\lambda) \rightarrow \mathbb{Q}$ such that: if \mathbf{Q} and $\mathbf{Q} + \tau$ are patches that occur in any tiling \mathbf{T} in Ω_ϕ and \mathbf{Q} has a vertex x_0 with $[x_0 - L', x_0 + L']$ contained in the support of \mathbf{Q} , then the number of occurrences of \mathbf{P} in \mathbf{T} between x_0 and $x_0 + \tau$ is exactly $\mathbf{N}_\mathbf{P}(\tau/L)$. That is, if x_1 is a vertex of \mathbf{P} , then $\{t : 0 \leq t < \tau \text{ and } \mathbf{P} + x_0 - x_1 + t \subset \mathbf{T}\}$ has cardinality $\mathbf{N}_\mathbf{P}(\tau/L)$.

¹ The tile and return lattices for a ‘proper’ substitution are the same, and a power of any primitive substitution can be ‘rewritten’ to a proper substitution - see *e.g.* [BD1].

Theorem 4 (Exact Regularity). *Homological Pisot substitution spaces exhibit the Exact Regularity Property. Moreover, given any patch \mathbf{P} , if $\mathbf{N}_{\mathbf{P}}$ is expressed in the form $\mathbf{N}_{\mathbf{P}}(\sum_{i=0}^{d-1} c_i \lambda^i) = \sum_{i=0}^{d-1} \alpha_i c_i$, with $\alpha_i, c_i \in \mathbb{Q}$, then $\alpha_i \in \mathbb{Z}[1/a_0]$ for $i = 0, \dots, d-1$, where a_0 is the constant coefficient of λ .*

Theorem 5. *Any Pisot substitution that exhibits the Exact Regularity Property is a homological Pisot substitution.*

Using the ERP, we derive restrictions on the measures of measurable sets in the tiling space.

Theorem 6. *Let ϕ be a homological Pisot substitution of algebraic degree d and constant coefficient a_0 . Suppose that the tile lattice and the return lattice for ϕ are the same. If some finite disjoint union of cylinder sets in the tiling space Ω_ϕ has rational measure n/m , with n and m relatively prime, then m divides $d(a_0)^k$ for some positive integer k .*

If $d = 1$, it is possible to construct a disjoint union of cylinder sets of measure $1/cr$ (Theorem 12). Theorem 6 then implies Theorem 2.

In Section 2, we develop necessary background and notation. In Section 3, we explore the ERP and prove Theorems 4, 5 and 6. The proofs of these theorems do not use the fact that the dilatation is a Pisot number, and the results of Section 3 apply to all substitutions for which the dimension of the first Čech cohomology equals the algebraic degree of the dilatation.

It is even possible to extend the idea of the ERP to one-dimensional tiling spaces that do not come from a substitution. If the first rational Čech cohomology of a tiling space Ω is k -dimensional, then we can find k different collections of patches (call them patches of type 1, 2, \dots , k), such that for any other patch \mathbf{P} there is a length L' with the following property: If \mathbf{Q} and $\mathbf{Q} + \tau$ are patches that occur in any tiling \mathbf{T} in Ω and \mathbf{Q} has a vertex x_0 with $[x_0 - L', x_0 + L']$ contained in the support of \mathbf{Q} , then the number of occurrences of \mathbf{P} in \mathbf{T} between x_0 and $x_0 + \tau$ is a rational linear function of n_1, \dots, n_k , where n_i is the number of occurrences of patches of type i between x_0 and $x_0 + \tau$. When the tiling space comes from a substitution and k is the algebraic degree of the dilatation, then the i -th class of patches is associated with tiles of length $L\lambda^{i-1}$.

In Section 4, we study $d = 1$ homological Pisot substitutions and prove Theorem 2. We also show that a homological Pisot substitution with $d = 1$ cannot have $cr = 2$, regardless of the norm. Finally, in Section 5 we show how to construct homological Pisot substitutions of arbitrary d with $cr = 3$, thereby proving Theorem 1.

2. BACKGROUND

A *substitution* is a function $\phi : \mathcal{A} \rightarrow \mathcal{A}^*$ from a finite *alphabet* \mathcal{A} into the collection \mathcal{A}^* of finite nonempty words in \mathcal{A} . A substitution extends by concatenation to a map on finite or infinite words and can be iterated: ϕ^m will stand for $\phi \circ \phi \circ \cdots \circ \phi$, m times. The *abelianization* or *substitution matrix* of ϕ is the matrix $\mathbf{A} = \mathbf{A}_\phi$ with ij -th entry equal to the number of i 's in $\phi(j)$. The substitution ϕ is *primitive* if the entries of \mathbf{A}^m are strictly positive for some $m \geq 1$. In this case \mathbf{A} has a simple, positive, Perron-Frobenius eigenvalue $\lambda = \lambda_\phi$ which we will call the *dilatation* of ϕ . If ϕ is primitive, we will use ω^l and ω^r to denote left and right positive eigenvectors of \mathbf{A} .

To construct the *tiling space* associated with the primitive substitution ϕ we first form the collection $\mathbf{X}_\phi \subset \mathcal{A}^\mathbb{Z}$ consisting of all bi-infinite *allowed words* for ϕ : $\bar{w} = \dots w_{-1}w_0w_1\dots \in \mathbf{X}_\phi$ if and only if for each $i \in \mathbb{Z}$, and each $j \geq 0$, there is an $m \in \mathbb{N}$ and $a \in \mathcal{A}$ so that the word $w_i \dots w_{i+j}$ is a factor (subword) of $\phi^m(a)$. If $\mathcal{A} = \{1, \dots, n\}$ and the left eigenvector of \mathbf{A} is $\omega^l = (\omega_1, \dots, \omega_n)$, the intervals $P_i = [0, \omega_i]$, $i = 1, \dots, n$, are called *prototiles* (in case $\omega_i = \omega_j$ for $i \neq j$, we label P_i and P_j so as to make them distinct). The tiling space, Ω_ϕ , associated with ϕ is the collection of all tilings of \mathbb{R} by translates of prototiles following patterns of allowed bi-infinite words.

A *patch* \mathbf{P} of a tiling \mathbf{T} is a subcollection of contiguous tiles of \mathbf{T} and the *support* of a patch is the union of all the tiles in the patch. We will denote the diameter of the support of \mathbf{P} by $|\mathbf{P}|$ and call this the *length* of \mathbf{P} .

There is a natural *tiling flow*, $\mathbf{T} = \{T_i\} \mapsto \mathbf{T} - t := \{T_i - t\}$. We put a metric d on Ω_ϕ with the property that \mathbf{T} and \mathbf{T}' are close if small translates of \mathbf{T} and \mathbf{T}' agree in a large neighborhood of the origin. If ϕ is primitive then, with this metric, Ω_ϕ is a continuum (compact and connected) and the tiling flow is minimal and uniquely ergodic.

The substitution ϕ is *aperiodic* provided there are no flow-periodic tilings in Ω_ϕ . There is also a \mathbb{Z} -action on Ω_ϕ induced by substitution. To define this, suppose that $i \in \mathcal{A}$ and $\phi(i) = i_1 \cdots i_k$. Define Φ on prototiles by

$$\Phi(P_i) := \{P_{i_1}, P_{i_2} + \omega_{i_1}, \dots, P_{i_k} + \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_{k-1}}\}.$$

Extend this to tiles by $\Phi(P_i + t) := \Phi(P_i) + \lambda t$, and extend to tilings by $\Phi(\{T_i\}) := \cup \Phi(T_i)$. As long as ϕ is primitive and aperiodic, Φ is a homeomorphism, see [M, So].

A *Pisot number* is an algebraic integer greater than 1, all of whose algebraic conjugates lie strictly inside the unit circle. We also recall the machinery of pattern-equivariant cohomology with rational coefficients from [Kel, KP, Sa]. A tiling $\mathbf{T} \in \Omega_\phi$ gives the real line the structure of a CW complex, with the vertices serving as 0-cells and the tiles serving as 1-cells.

Definition 7. A rational 0-cochain is said to be pattern-equivariant with radius R if, whenever x and y are vertices of \mathbf{T} and the radius R neighborhoods $\mathbf{B}_R[\mathbf{T}-x] = \mathbf{B}_R[\mathbf{T}-y]$, the cochain takes the same values at x and y . A 0-cochain is pattern-equivariant if it is pattern-equivariant with radius R for some finite R . Pattern-equivariant 1-cochains are defined similarly – their values on a 1-cell depend only on the pattern of the tiling out to a fixed finite distance around that 1-cell.

If β is a rational pattern-equivariant 0-cochain, its coboundary, $\delta(\beta)$, is a rational pattern-equivariant 1-cochain, and we define the rational first pattern-equivariant cohomology of \mathbf{T} to be the cokernel of the coboundary map δ . A priori this would seem to depend on \mathbf{T} , but this cohomology is the same for all $\mathbf{T} \in \Omega_\phi$ and is isomorphic to $\check{H}^1(\Omega_\phi, \mathbb{Q})$, see [Kel, KP, Sa].

Finally, we recall a procedure for computing the first Čech cohomology of a tiling space, using the machinery of [BD3]. To each primitive one-dimensional substitution ϕ on n letters, one can associate a graph G . This graph has two kinds of edges. There is one edge e_i for each letter a_i of the alphabet, and one edge v_{ij} for each two-letter word $a_i a_j$. The edge e_i has length $\omega_i - \epsilon$ and represents the bulk of a tile of type a_i , while v_{ij} has length ϵ and represents the transition from the end of a tile of type i to the beginning of a tile of type j . We identify the end of e_i with the beginning of v_{ij} , and the end of v_{ij} with the beginning of e_j . There is also a sub-complex G_0 obtained from just the v edges.

After applying a small homotopy, we may assume that substitution maps G_0 to itself. If $\phi(a_i)$ ends with a_k , and if $\phi(a_j)$ begins with a_ℓ , then $\phi(v_{ij}) = v_{k\ell}$. Let G_0^{ER} be the eventual range of G_0 under this map. Since substitution permutes the edges of G_0^{ER} , we can replace ϕ with a power that fixes each edge of G_0^{ER} . Then there is an exact sequence, [BD3]

$$(1) \quad 0 \rightarrow \check{H}^0(G_0^{ER}) \rightarrow \varinjlim \mathbf{A}^T \rightarrow \check{H}^1(\Omega_\phi) \rightarrow H^1(G_0^{ER}) \rightarrow 0$$

that computes $\check{H}^1(\Omega_\phi)$. Furthermore, each map in this exact sequence commutes with substitution, so the image of $\check{H}^0(G_0^{ER})$ lies in the $+1$ eigenspace of \mathbf{A}^T . Since the dilatation and its algebraic conjugates are all eigenvalues of \mathbf{A}^T , the dimension of $\check{H}^1(\Omega_\phi, \mathbb{Q})$ is at least d , and equals d only if three conditions are met: (1) the only eigenvalues of \mathbf{A}^T are 0, 1, λ , and the algebraic conjugates of λ , (2) the algebraic multiplicity of the eigenvalue 1 is one less than the number of components of G_0^{ER} , and (3) G_0^{ER} has no loops.

3. EXACT REGULARITY

Given a substitution ϕ with dilatation λ of degree d , let us select a return length L . Then for every (by minimality of the flow) $\mathbf{T} \in \Omega_\phi$, \mathbf{T} and $\mathbf{T} - L$ have nonempty

intersection. It follows that for any L' , there exists a k such that $\phi^k(\mathbf{T})$ and $\phi^k(\mathbf{T}) - \lambda^k L$, and hence \mathbf{T} and $\mathbf{T} - \lambda^k L$, intersect in a patch of length at least L' .

As $1, \lambda, \lambda^2, \dots, \lambda^{d-1}$ are linearly independent over \mathbb{Q} , the length of each patch \mathbf{P} can be expressed uniquely in the form $|\mathbf{P}| = L \sum_{i=0}^{d-1} c_i \lambda^i$ with $c_i = c_i(|\mathbf{P}|) \in \mathbb{Q}$. Given any tiling $\mathbf{T} \in \Omega_\phi$, let ξ_i be the pattern equivariant 1-cochain on \mathbf{T} defined by $\xi_i(T) := c_i(|T|)$ for each $T \in \mathbf{T}$.

Lemma 8. *The cohomology classes $[\xi_0], \dots, [\xi_{d-1}]$ are linearly independent in the pattern-equivariant cohomology of Ω_ϕ .*

Proof. Suppose that $\alpha = \sum \alpha_i \xi_i = \delta(\beta)$, where β is a pattern-equivariant 0-cochain and each α_i is rational. For large enough k , $\lambda^k L$ will be a return length between patches of size greater than twice the radius of β . Therefore, α applied to $\lambda^{k+i} L$ must be zero for $i \geq 0$. As $\lambda^d + a_{d-1} \lambda^{d-1} + \dots + a_0 = 0$, after division by λ , we have $a_0 \lambda^{-1} = -(\lambda^{d-1} + a_{d-1} \lambda^{d-2} + \dots + a_1)$. Taking the $(k-i)$ th power of the last equation, we have $a_0^{k-i} \lambda^{i-k}$ as a polynomial in λ with integer coefficients. Hence, $a_0^{k-i} L \lambda^i$ is an integer linear combination of $\lambda^k L, \lambda^{k+1} L, \dots, \lambda^{k+d-1} L$, and α_i must be zero. \square

If ϕ is a homological Pisot substitution, $\{[\xi_0], \dots, [\xi_{d-1}]\}$ is a basis for the first cohomology of any tiling in Ω_ϕ . We use this to prove the Exact Regularity Property (Theorem 4) and its converse (Theorem 5).

Proof of Theorem 4. Let α be a pattern-equivariant 1-cochain on \mathbf{T} (which we call an *indicator cochain*) that evaluates to 1 at each occurrence of the patch \mathbf{P} and is identically zero away from \mathbf{P} . That is, choose a tile $T \in \mathbf{P}$ and define α by: if T' is any tile of \mathbf{T} , then $\alpha(T') = 1$ if there is a $t \in \mathbb{R}$ such that $T + t = T'$ and $\mathbf{P} + t \subset \mathbf{T}$, and $\alpha(T') = 0$ otherwise. We must then have $\alpha = \delta(\beta) + \sum \alpha_i \xi_i$, where β is a pattern-equivariant 0-cochain of some radius r and the coefficients α_i are rational. Let $L' = \max\{r, |\mathbf{P}|\}$ and suppose that patch $\mathbf{Q} \subset \mathbf{T}$ has a vertex x_0 with $[x_0 - L', x_0 + L']$ contained in the support of \mathbf{Q} . Suppose also that $\mathbf{Q} + \tau \subset \mathbf{T}$. Since $\beta(x_0 + \tau) = \beta(x_0)$, $\alpha([x_0, x_0 + \tau]) = \sum \alpha_i \xi_i([x_0, x_0 + \tau]) = \sum \alpha_i c_i(\tau)$. On the other hand, $\alpha([x_0, x_0 + \tau])$ is the number of occurrences of \mathbf{P} between x_0 and $x_0 + \tau$, this number being unambiguous (*i.e.*, independent of choice of $T \in \mathbf{P}$) since $L' \geq |\mathbf{P}|$.

If L'' is any return length, then for large k , $\lambda^{k+i} L''$ is a return length between patches of size greater than L' for all $i \geq 0$. Then $\sum \alpha_i \xi_i = \alpha - \delta(\beta)$ applied to $\lambda^{k+i} L''$ yields an integer. However, $a_0^k L''$ is an integer linear combination of $\lambda^k L'', \lambda^{k+1} L'', \dots, \lambda^{k+d-1} L''$. This implies that $\sum \alpha_i \xi_i$ applied to L'' yields an integer divided by a power of a_0 . Since $\sum \alpha_i \xi_i$ applied to any element of the return lattice yields an integer divided by a power of a_0 , and since $L \lambda^i$ is in the return lattice, all coefficients α_i must be integers divided by powers of a_0 . \square

Proof of Theorem 5. Let α be an indicator cochain for a patch \mathbf{P} (as in the proof above). There are then $\alpha_i \in \mathbb{Q}$ so that $\gamma := \alpha - \sum \alpha_i \xi_i$ vanishes on chains of the form $[x_0, x_0 + \tau]$ with x_0 a vertex in a patch \mathbf{Q} of \mathbf{T} , $[x_0 - L', x_0 + L']$ contained in the support of \mathbf{Q} , and $\mathbf{Q} + \tau \subset \mathbf{T}$. Let \mathbf{Q} and x_0 be such a patch and vertex and define the 0-cochain g by $g(x) := \gamma([x_0, x])$ for each vertex x of \mathbf{T} (here $[x_0, x]$ means $-[x, x_0]$ for $x < x_0$, and we take $g(x_0) = 0$). Then g is pattern-equivariant and $\delta g = \gamma$. Since the indicator cochains span the pattern-equivariant 1-cochains, the $[\xi_i]$ form a basis for the first cohomology of \mathbf{T} . \square

Having established constraints on the number of occurrences of \mathbf{P} for any return length L (or at least $\lambda^k L$ for sufficiently large k), we establish constraints on the measure of the cylinder set of \mathbf{P} with respect to the unique translation-invariant measure μ on Ω_ϕ . Let $S_{\mathbf{P}}$ be the set of all tilings in which the origin lies inside a \mathbf{P} patch.

Theorem 9. *Suppose that ϕ is a homological Pisot substitution of degree d for which the tile and return lattices are the same, and let $p'(\lambda) = d\lambda^{d-1} + \sum_{i=1}^{d-1} ia_i\lambda^{i-1}$ be the derivative of the minimal polynomial of the dilatation λ of ϕ , evaluated at λ . Then $\mu(S_{\mathbf{P}}) = \frac{q_{\mathbf{P}}(\lambda)}{a_0^k p'(\lambda)}$, where k is an integer and $q_{\mathbf{P}}(\lambda)$ is a polynomial in λ with integer coefficients.*

Proof. Let \mathbf{P} be a patch based on the word $w = w_1 \cdots w_l$. Given any patch \mathbf{Q} , let $S_{\mathbf{Q}}^j$ denote the (partial) cylinder set consisting of all tilings for which the origin not only lies in a \mathbf{Q} patch, but in the j th tile of the \mathbf{Q} patch. We may then express $S_{\mathbf{P}}$ as a (measurably) disjoint union of sets of the form $S_{\mathbf{Q}}^{l+1}$ where \mathbf{Q} has underlying word $uw_i v$, u and v l -letter words, $i \in \{1, \dots, l\}$. For \mathbf{Q} based on the word $uw_i v$ and m such that the length of $\phi^m(w_i)$ is a return length, we take $L = \lambda^m |w_i|$. To compute the measure of $S_{\mathbf{Q}}^{l+1}$, we merely count how many times \mathbf{Q} occurs in an interval of length $L\lambda^k$, divide by λ^k , take the limit as $k \rightarrow \infty$, and then multiply by $|w_i|/L$. To find the limit, we write $\lambda^k = \sum_{i=0}^{d-1} c_{k,i} \lambda^i$ and take the limit of $\sum \alpha_i c_{k,i} / \lambda^k$. The limit of $\vec{r} = \lim_{k \rightarrow \infty} (c_{k,0}, \dots, c_{k,d-1})^T / \lambda^k$ is a right eigenvector of the companion matrix

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix},$$

normalized so that $(1, \lambda, \dots, \lambda^{d-1})\vec{r} = 1$. This eigenvector is

$$\vec{r} = \frac{1}{p'(\lambda)} \begin{pmatrix} \lambda^{d-1} + a_{d-1}\lambda^{d-2} + \dots + a_1 \\ \lambda^{d-2} + a_{d-1}\lambda^{d-3} + \dots + a_2 \\ \lambda^{d-3} + a_{d-1}\lambda^{d-4} + \dots + a_3 \\ \vdots \\ \lambda + a_{d-1} \\ 1 \end{pmatrix}.$$

Since each entry of \vec{r} is a polynomial in λ divided by $p'(\lambda)$, and since each α_i is an integer divided by a power of a_0 , the measure of $S_{\mathbf{Q}}^{l+1}$ is of the indicated form. It follows that $S_{\mathbf{P}}$ has the desired form as well. \square

Theorem 10. *Let ϕ be a homological Pisot substitution of degree d and dilatation λ for which the tile and return lattices are the same. If some cylinder set in Ω_ϕ has rational measure n/m , with n and m relatively prime, then m divides $a_0^k \gcd(p'(\lambda))$ for some $k \in \mathbb{N}$, where $\gcd(p'(\lambda))$ is the greatest common divisor of the coefficients of $p'(\lambda)$ and a_0 is the constant coefficient of λ .*

Proof. If $\frac{q_{\mathbf{P}}(\lambda)}{a_0^k p'(\lambda)} = \frac{n}{m}$, then $m q_{\mathbf{P}}(\lambda) = a_0^k p'(\lambda) n$. But this means that m divides every coefficient of $a_0^k p'(\lambda)$, and so divides $a_0^k \gcd(p'(\lambda))$. \square

Theorem 6 is an immediate corollary of Theorem 10.

4. SUBSTITUTIONS WITH $d = 1$

If $d = 1$, then the dilatation is an integer N , and the norm is N itself. Under these circumstances, we can replace the substitution with an ‘equivalent’ substitution that has constant length using the following technique.

First scale the left Perron-Frobenius eigenvector so that all entries are integers whose greatest common factor is one. That is, choose all of the tiles to have integral length. Then subdivide each tile into smaller pieces, each of length one. The substitution, written in terms of these pieces, will have constant length N . If the resulting substitution has return lattice $h\mathbb{Z}$, with $h > 1$, we can regroup these pieces into tiles of size h and rescale by h . The resulting substitution has constant length, with both the tile and return lattices equaling \mathbb{Z} , and is called the “pure core” of the original substitution. This new substitution and the original substitution have conjugate tiling flows, hence their coincidence ranks are the same and one is homological Pisot if and only if the other is.

Consider the substitution ϕ of constant length N acting on the alphabet \mathcal{A}_ϕ . We say that two words $w_1 w_2 w_3 \dots$ and $v_1 v_2 v_3 \dots$ are *coincident* if there exists a k such that

$w_k = v_k$. We say that the letters a and b in \mathcal{A}_ϕ are *eventually coincident* if there exist a k and n such that $\phi^n(a)_k = \phi^n(b)_k$, where $\phi^n(a)_k$ denotes the k th letter of $\phi^n(a)$. We say that the letters a and b are *strongly coincident* if for each $n \geq 0$ and each $i \in \{1, \dots, N^n\}$, the i th letters of $\phi^n(a)$ and $\phi^n(b)$ are eventually coincident.

Theorem 11. *Let ϕ be a substitution of constant length N and let ϕ' be the substitution obtained by identifying the letters in \mathcal{A}_ϕ that are strongly coincident. If ϕ is a homological Pisot substitution, then so is ϕ' .*

Proof. If ϕ is a homological Pisot substitution, then ϕ has the Exact Regularity Property by Theorem 4. Since every patch \mathbf{P}' in a tiling of $\Omega_{\phi'}$ corresponds to a finite set of patches $\{\mathbf{P}_1, \dots, \mathbf{P}_n\}$ in Ω_ϕ , and since each of these patches \mathbf{P}_i is governed by the ERP, we will show that \mathbf{P}' is governed by the ERP, implying that ϕ' is a homological Pisot substitution.

The patches \mathbf{P}_i exhibit the ERP with different lengths L'_i . Let $L' = \max\{L'_i\}$. Pick any patch \mathbf{S}' in $\Omega_{\phi'}$ with a vertex that is a distance at least L' from each end. \mathbf{S}' corresponds to a finite set $\{\mathbf{S}_1, \dots, \mathbf{S}_m\}$ of patches in Ω_ϕ , each with a vertex of distance at least L' from the end. By the ERP for each \mathbf{P}_i , between any two successive \mathbf{S}_i 's, there are exactly the right number of \mathbf{P}_1 's, the right number of \mathbf{P}_2 's, etc, so between any two \mathbf{S}' 's that correspond to the same \mathbf{S}_i , there are exactly the right number of \mathbf{P}' 's. The problem is that different occurrences of \mathbf{S}' may correspond to different \mathbf{S}_i 's.

The discrepancy in how many extra \mathbf{P}' 's occur between an \mathbf{S}'_i and an \mathbf{S}'_j depends only on i and j , since the discrepancy between any two \mathbf{S}'_i 's, or any two \mathbf{S}'_j 's, is zero. In particular, there exist numbers z_1, \dots, z_m so that there are exactly $z_i - z_j$ extra \mathbf{P}' 's between any \mathbf{S}'_i and any \mathbf{S}'_j . Order the images such that $z_1 \geq z_2 \geq \dots \geq z_m$. Since ϕ is a primitive substitution, Ω_ϕ is repetitive, so every sufficiently large patch (say, of diameter D) contains at least one copy of \mathbf{S}_1 and at least one copy of \mathbf{S}_m .

Now let \mathbf{Q}' be any patch of size greater than D . Each patch \mathbf{Q}_i in Ω_ϕ that maps to \mathbf{Q}' must contain an \mathbf{S}_1 and an \mathbf{S}_m . The images of the \mathbf{S} patches with maximal and minimal values of z_i can be identified, in \mathbf{Q}' , by the $z_1 - z_m$ extra (or missing) \mathbf{P}' 's that occur between them, which can only occur between an \mathbf{S}'_i with $z_i = z_1$ and an \mathbf{S}'_j with $z_j = z_m$. This means that the locations of patches \mathbf{S}'_i with $z_i = z_1$ line up exactly for any two patches \mathbf{Q}' , so the number of \mathbf{P}' 's between any two \mathbf{Q}' 's is exactly correct. \square

We define a *stable pair* as two letters in \mathcal{A}_ϕ which are not eventually coincident and extend this idea to arbitrary collections of letters: a *stable m -tuple* is a collection $\{a_1, a_2, \dots, a_m\}$ of letters from \mathcal{A}_ϕ such that for all n and $0 \leq k \leq N^n$, $\{\phi^n(a_1)_k, \phi^n(a_2)_k, \dots, \phi^n(a_m)_k\}$ has cardinality m . For constant length substitutions, the coincidence rank, cr , is the minimum cardinality of $\{\phi^n(a_1)_k, \dots, \phi^n(a_{|\mathcal{A}_\phi|})_k\}$ for $0 \leq k \leq N^n, n \in \mathbb{Z}$, see [BBK]. Hence $m = cr$ is the maximal cardinality of a stable

m -tuple and it follows that the coincidence ranks of ϕ and ϕ' are the same. Therefore, to obtain constraints on cr for homological Pisot substitutions, it suffices to consider substitutions for which no two letters are strongly coincident.

Theorem 12. *If ϕ is a homological Pisot substitution of constant length N and coincidence rank cr , with no strongly coincident pairs of letters, then there exists a set of letters whose cylinder sets have total measure $1/cr$.*

Proof. For sufficiently large n , there exists a k such that $\{\phi^n(a_i)_k\}$, $i \in \{1, \dots, |\mathcal{A}_\phi|\}$ consists of exactly cr elements. (Asymptotically, the number of such k , divided by N^n , approaches 1, [De]). Let n and k be as described above, and let the set $\{\phi^n(a_i)_k\}$ be given by $\{b_1, b_2, \dots, b_{cr}\}$. This set is a stable cr -tuple. There are then tilings $\mathbf{T}^i \in \Omega_\phi$, $i = 1, \dots, cr$, and an $m \in \mathbb{N}$, so that $\Phi^m(\mathbf{T}^i) = \mathbf{T}^i$ and 0 is the left endpoint of a tile of type b_i in \mathbf{T}^i . Partition \mathcal{A}_ϕ into sets $B_j := \{a : \phi^n(a)_k = b_j\}$, $j \in \{1, \dots, cr\}$. Let C_j be the union of the cylinder sets corresponding to letters in B_j ; $C_j := \{\mathbf{T} : 0 \in T_0, T_0 \in \mathbf{T}, T_0 \text{ of type } a \text{ for some } a \in B_j\}$. For fixed j and for each $t \in \mathbb{R}^+ \setminus \mathbb{N}$, exactly one of the tilings \mathbf{T}^i is in C_j . Thus $1 = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \sum_{j=1}^{cr} \chi_{C_j}(\mathbf{T}^j - t) dt = \sum_{j=1}^{cr} \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \chi_{C_j}(\mathbf{T}^j - t) dt = cr\mu(C_j)$, by unique ergodicity of the tiling flow. Thus, for each j , the measure of the union of the cylinder sets of the elements of B_j is $1/cr$. \square

By Theorem 6, this implies that cr divides a power of N , thereby proving Theorem 2, namely that the Coincidence Rank Conjecture holds for $d = 1$.

Theorem 13. *If ϕ is a homological Pisot substitution with $d = 1$, then the coincidence rank is not 2.*

Proof. If N is odd, this follows from Theorem 2. Suppose, then, that N is even and $cr = 2$. We will prove that $\dim(H^1) > 1$. Every element of \mathcal{A}_ϕ must be a member of a stable pair by primitivity and the fact that there exist n, k such that $\{\phi^n(a_1)_k, \phi^n(a_2)_k, \dots, \phi^n(a_{|\mathcal{A}_\phi|})_k\}$ has cardinality cr . Suppose that a is in a stable pair with a' and a'' , $a' \neq a''$. Then a' and a'' must be strongly coincident, or $cr = 2$ is violated. By identifying strongly coincident letters, and recalling the results of Theorem 11, it suffices to consider the case of an alphabet of $2m$ letters $\{a_1, \dots, a_m, a'_1, \dots, a'_m\}$, where a_i and a'_i form a stable pair.

Ordering the columns and rows of the abelianization by $a_1, \dots, a_m, a'_1, \dots, a'_m$, the matrix takes the form $\mathbf{A} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$, where A and B are $m \times m$ matrices. In particular, the trace of \mathbf{A} is even.

The space G_0^{ER} appearing in (1) is the direct limit under substitution of the graph obtained from all the edges v_{ij} that describe two-letter words $a_i a_j$ and likewise edges $v_{i'j}$, $v_{ij'}$ and $v_{i'j'}$.

The symmetry that exchanges each a_i for a'_i maps the components of G_0^{ER} to each other. If any component is mapped to itself, then there exists a loop in G_0^{ER} , implying that the dimension of $\check{H}^1(\Omega_\phi)$ exceeds d . To see this, suppose that v_{ij} and $v_{i'j'}$ are in the same component. Then there exists a path in the graph connecting these two edges. Letting $v_{ij} = v_{i_1j_1}$, this path may be labeled as $v_{i_1j_1}, v_{i_2j_1}, v_{i_2j_2}, v_{i_3j_2}, \dots, v_{i'j'}$. Since the existence of the word $a_i a_j$ in the substitution implies the appearance of $a'_i a'_j$ as well, the path $v_{i'j'} = v_{i'_1j'_1}, v_{i'_2j'_1}, v_{i'_2j'_2}, \dots, v_{ij}$ exists also, creating a loop in G_0^{ER} .

If no component is mapped to itself, then there are an even number of components in G_0^{ER} , which come in symmetric pairs. Since ϕ is a homological Pisot substitution, the dimension of $\check{H}^1(\Omega_\phi)$ is one. This implies that the nonzero eigenvalues of \mathbf{A}^T must be N (with multiplicity one) and 1 (with odd multiplicity). This is impossible, since N is even and the trace of \mathbf{A}^T is even. \square

5. EXAMPLES

In this section we provide examples of homological Pisot substitutions with $cr > 1$. Specifically, all examples have $cr = 3$ and all dilatations have norm divisible by 3. In each case we first construct a homological Pisot substitution with $cr = 1$ and alphabet $\mathcal{A}_1 = \{A, B, \dots\}$, and then construct another homological Pisot substitution with alphabet $\mathcal{A}_2 = \{a_1, a_2, a_3, b_1, b_2, b_3, \dots\}$ that is a triple cover of the first, and hence has $cr = 3$.

Example 1: $d = 1$

We take a 2-letter alphabet $\mathcal{A}_1 = \{A, B\}$ and a substitution

$$\begin{aligned} \phi_1(A) &= ABABAAABA \\ \phi_1(B) &= BAAABAABA \end{aligned} \quad \text{with abelianization} \quad \begin{pmatrix} 6 & 3 \\ 6 & 3 \end{pmatrix}$$

Since all substituted letters end in A , the first cohomology of Ω_{ϕ_1} is the direct limit of the abelianization (see [BD2]) and has dimension one.

For \mathcal{A}_2 , we associate three letters to each letter in \mathcal{A}_1 , and a permutation of those three letters to each transition AA , AB , BA or BB . (Actually, there are no transitions BB , but we include a permutation as a demonstration of the method.) Specifically, to the transition AA we associate $a_1 a_3$, $a_3 a_1$ and $a_2 a_2$. To AB we associate $a_1 b_2$, $a_2 b_1$ and $a_3 b_3$. To BA we associate $b_1 a_3$, $b_3 a_1$ and $b_2 a_2$. To BB we associate $b_1 b_2$, $b_2 b_1$, and $b_3 b_3$. In short, we are allowing transitions $\{a_1 \text{ or } b_1\}\{a_3 \text{ or } b_2\}$, $\{a_2 \text{ or } b_2\}\{a_2 \text{ or } b_1\}$, and $\{a_3 \text{ or } b_3\}\{a_1 \text{ or } b_3\}$.

Note that there are exactly three words in the alphabet \mathcal{A}_2 associated to each word in the alphabet \mathcal{A}_1 . There are three choices on what the first letter should be, and the rest of the word is determined by the allowed transitions. These three words disagree at each point.

The substitution that generates these words is

$$(2) \quad \begin{aligned} \phi_2(a_1) &= a_1 b_2 a_2 b_1 a_3 a_1 a_3 b_3 a_1 \\ \phi_2(a_2) &= a_2 b_1 a_3 b_3 a_1 a_3 a_1 b_2 a_2 \\ \phi_2(a_3) &= a_3 b_3 a_1 b_2 a_2 a_2 b_1 a_3 \\ \phi_2(b_1) &= b_1 a_3 a_1 a_3 b_3 a_1 a_3 b_3 a_1 \\ \phi_2(b_2) &= b_2 a_2 a_2 a_2 b_1 a_3 a_1 b_2 a_2 \\ \phi_2(b_3) &= b_3 a_1 a_3 a_1 b_2 a_2 a_2 b_1 a_3 \end{aligned} \quad \text{with abelianization} \quad \begin{pmatrix} 3 & 2 & 1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 0 & 4 & 2 \\ 2 & 2 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \end{pmatrix}$$

Note that for each x_i (where $x = a$ or b), $\phi(x_i)$ begins with x_i and ends with a_i . This ensures that substituted words will have the same allowed transitions as the original words.

The abelianization has eigenvalues $9, 1, 1, 0, 0, 0$. Since the direct limit of the transition graph has 3 components, each of which is contractible, this implies that $\check{H}^1(\Omega_\phi, \mathbb{Q})$ is 1-dimensional.

Example 2: $d = 2$

We take \mathcal{A}_1 and \mathcal{A}_2 as before, only now ϕ_1 is a substitution of degree 2, with abelianization $\begin{pmatrix} 9 & 6 \\ 6 & 3 \end{pmatrix}$ and dilatation $6 + 3\sqrt{5}$:

$$\begin{aligned} \phi_1(A) &= ABABAAABABABABA \\ \phi_1(B) &= BAAABAABA \end{aligned}$$

This is just like the $d = 1$ example, only with a $BABABA$ suffix applied to $\phi_1(A)$. Construct ϕ_2 similar to (2). This suffix induces a trivial permutation, and so continues to allow us to have $\phi_2(x_i)$ begin with x_i and end with a_i . The abelianization for ϕ_2 is now

$$\begin{pmatrix} 4 & 3 & 2 & 3 & 1 & 2 \\ 2 & 3 & 4 & 0 & 4 & 2 \\ 3 & 3 & 3 & 3 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 2 & 0 & 1 \end{pmatrix},$$

and has eigenvalues $6 + 3\sqrt{5}, 6 - 3\sqrt{5}, 1, 1, 0, 0$. As before, the direct limit of the transition graph has 3 components, each of which is contractible, so $\check{H}^1(\Omega_\phi, \mathbb{Q})$ is 2-dimensional.

Example 3: $d = 3$

We now take $\mathcal{A}_1 = \{A, B, C\}$, with substitution

$$\begin{aligned} \phi_1(A) &= ABABAAABA(CBA)^6(BA^3)^3 \\ \phi_1(B) &= BAAABAABA(CBA)^3(CBA^3)^3 \\ \phi_1(C) &= (CBA)^3(BA^3)^3. \end{aligned}$$

with dilatation $\lambda = 3\theta^4$ where θ is the tribonacci dilatation, *i.e.*, the leading root of $\theta^3 = \theta^2 + \theta + 1$. The eigenvalues of the abelianization of ϕ_1 are λ and its algebraic conjugates. It is not hard to see that $\dim(\check{H}^1(\Omega_\phi, \mathbb{Q})) = 3$, and that the coincidence rank of ϕ_1 is 1.

As far as ϕ_2 and permutations of $\{1, 2, 3\}$ go, C is merely a spectator, with transitions ending in C having a trivial permutation and transitions beginning with c_i having the same permutations as transitions beginning with a_i or b_i . Specifically, the allowed transitions for Ω_{ϕ_2} are $\{a_1 \text{ or } b_1 \text{ or } c_1\}\{a_3 \text{ or } b_2 \text{ or } c_1\}$, $\{a_2 \text{ or } b_2 \text{ or } c_2\}\{a_2 \text{ or } b_1 \text{ or } c_2\}$, and $\{a_3 \text{ or } b_3 \text{ or } c_3\}\{a_1 \text{ or } b_3 \text{ or } c_3\}$.

The terms $(CBA)^3$, $(BA^3)^3$ and $(CBA^3)^3$ induce trivial permutations of $\{1, 2, 3\}$ and give equal populations of x_1, x_2, x_3 to ϕ_2 of any letter. They are special cases of the “padding” discussed below in Example 4.

The nonzero eigenvalues of the abelianization of ϕ_2 are $1, 1, \lambda$, and the algebraic conjugates of λ . Since the eventual range of the transition graph of ϕ_2 has 3 contractible components, $\dim(\check{H}^1(\Omega_{\phi_2})) = \dim(\check{H}^1(\Omega_{\phi_1})) = 3$.

Example 4: Arbitrary d

The same tricks can be used to create examples of arbitrary degree and $cr = 3$.

Start with any primitive $d \times d$ Pisot matrix M_0 with odd determinant. Since the determinant is 1 (mod 2), some power M_0^k is equal to the identity (mod 2), and we can assume that k is big enough for all but one of the eigenvalues of M_0^k to be smaller than $1/3$. Let $M_1 = 3M_0^k$ be a new Pisot abelianization on d letters A, B, \dots, Z , which is the abelianization of a substitution of the form

$$\begin{aligned} \phi_1(A) &= ABABAAABA \text{ plus padding,} \\ \phi_1(B) &= BAAABAABA \text{ plus padding,} \\ \phi_1(\text{any other letter}) &= \text{nothing but padding,} \end{aligned}$$

where “padding” means a product of words, each of the general form $(VB^{\text{odd}}WA^{\text{odd}}Y)^3$, where V, W and Y are arbitrary words in the letters other than A and B . We can always choose the padding such that ϕ_1 of any letter begins with that letter and ends in A .

For Ω_{ϕ_2} , the allowed transitions are $\{\text{any } x_1\}\{a_3 \text{ or } b_2 \text{ or any other } y_1\}$, $\{\text{any } x_2\}\{a_2 \text{ or } b_1 \text{ or any other } y_2\}$, and $\{\text{any } x_3\}\{a_1 \text{ or } b_3 \text{ or any other } y_3\}$. With these choices, padding yields trivial permutations, and we can pick $\phi_2(x_i)$ to begin with x_i and end with a_i .

The abelianization M_2 for ϕ_2 can be viewed as a collection of 3×3 blocks, one for each matrix element of the abelianization for ϕ_1 . All blocks other than the AA, AB, BA and BB blocks are multiples of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. This is because “padding” yields equal

numbers of x_1 , x_2 and x_3 . The AA , AB , BA and BB blocks are the same as in our $d = 1$ example, plus multiples of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

The populations of $\phi_2(a_1)$ and $\phi_2(a_3)$ add up to twice the population of $\phi_2(a_2)$, and the populations of $\phi_2(b_1)$ and $\phi_2(b_2)$ add up to twice the population of $\phi_2(b_3)$. For $X \neq A$ or B , the populations of $\phi_2(x_1)$, $\phi_2(x_2)$ and $\phi_2(x_3)$ are all the same. This yields $2d - 2$ vectors in the kernel of M_2 , so the rank of M_2 is at most $d + 2$. However, every eigenvalue of M_1 is also an eigenvalue of M_2 . Since the eventual range of the transition graph of ϕ_2 has three components, 1 must be an eigenvalue of M_2 with multiplicity 2. Together, these imply that $\dim(\check{H}^1(\Omega_{\phi_2})) = d$.

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REFERENCES

- [A] J. Auslander, Minimal flows and their extensions, *North-Holland Mathematical Studies*, vol. 153, North-Holland, Amsterdam, New York, Oxford, and Tokyo, (1988).
- [AP] J. E. Anderson and I. F. Putnam, Topological invariants for substitution tilings and their associated C^* -algebras, *Ergodic Theory & Dynamical Systems* **18** (1998), 509–537.
- [BBK] V. Baker, M. Barge and J. Kwapisz, Geometric realization and coincidence for reducible non-unimodular Pisot tiling spaces with an application to β -shifts, *J. Instit. Fourier*. **56** (7) (2006), 2213–2248.
- [BD1] M. Barge and B. Diamond, A complete invariant for the topology of one-dimensional substitution tiling spaces, *Ergodic Theory & Dynamical Systems* **21** (2001), 1333–1358.
- [BD2] M. Barge and B. Diamond, Coincidence for substitutions of Pisot type, *Bulletin de la Société Mathématique de France* **130** (2002), 619–626.
- [BD3] M. Barge and B. Diamond, Cohomology in one-dimensional substitution tiling spaces, *Proc. Amer. Math. Soc.* **136** (6) (2008), 2183–2191.
- [BK] M. Barge and J. Kwapisz, Geometric theory of unimodular Pisot substitutions, *Amer. J. Math.* **128** (2006), 1219–1282.
- [BS] V. Berthé and A. Siegel, Tilings associated with beta-numeration and substitutions, *Integers: electronic journal of combinatorial number theory* **5** (2005), A02.
- [BSw] M. Barge and R. Swanson, Rigidity in one-dimensional tiling spaces, *Top. and its Appl.* **157** (17) (2007), 3095–3099.
- [BT] E. Bombieri and J. E. Taylor, Which distributions of matter diffract? An initial investigation, *J. Physique* **47** (7, suppl. Colloq. C3) C3-19–C3-28, International workshop on aperiodic crystals (les Houches) (1986).
- [CS] V. Canterini and A. Siegel, Geometric representation of substitutions of Pisot type, *Transactions of the American Mathematical Society* **353** (2001), 5121–5144.

- [De] F. M. Dekking, The spectrum of dynamical systems arising from substitutions of constant length, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **41** (1978), 221-239.
- [Dw] S. Dworkin, Spectral theory and x-ray diffraction, *J. Math. Phys.* **34** (1993), 2964-2967.
- [HS] M. Hollander and B. Solomyak, Two-symbol Pisot substitutions have pure discrete spectrum, *Ergodic Theory & Dynamical Systems* **23** (2003), 533-540.
- [LMS] J.-Y. Lee, R. V. Moody and B. Solomyak, Pure point dynamical and diffraction spectra, *Annales Henri Poincaré* **3** (2002), 1003-1018.
- [M] B. Mossé, Puissances de mots et reconnaissabilité des points fixes d'une substitution, *Theoretical Computer Science* **99** (1992), 327-334.
- [IR1] S. Ito and H. Rao, Atomic surfaces, tilings and coincidences I. Irreducible case, *Israel J. Math.* **153** (2006), 129-156.
- [IR2] S. Ito and H. Rao, Atomic surfaces, tilings and coincidences II. Reducible case, *Ann. Inst. Fourier, Grenoble* **56** (6) (2006), 2285-2313.
- [Kel] J. Kellendonk, Pattern-equivariant functions and cohomology, *J. Phys. A.* **36** (2003), 1-8.
- [KP] J. Kellendonk and I. Putnam, The Ruelle-Sullivan map for \mathbb{R}^n -actions, *Math. Ann.* **334** (2006), 693-711.
- [Sa] L. Sadun, Pattern-equivariant cohomology with integer coefficients. *Ergodic Theory & Dynamical Systems* **27** (2007), 1991-1998.
- [So] B. Solomyak, Nonperiodicity implies unique composition for self-similar translationally finite tilings, *Discrete and Computational Geometry* **20** (2) (1998), 265-279.

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